REMARKS ON THE ENTROPY OF NON-STATIONARY BLACK HOLES

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Abstract

The definition of entropy obtained for stationary black holes is extended in this paper to the case of non-stationary black holes. Entropy is defined as a macroscopical thermodynamical quantity which satisfies the first principle of thermodynamics. In the non-stationary case a volume term appears since the solution does not admit a Killing vector.

1 Introduction

It is known that in many situations the entropy of a black hole solution of Einstein's equations can be calculated in the framework of a classical field theory without resorting to a statistical approach.
This may be considered a good feature at present since statistical approaches are based on a Hamiltonian and/or quantum formulation and both these aspects are not yet clear for General Relativity. The definition of black hole entropy for stationary black holes as a macroscopical quantity which satisfies the first principle of thermodynamics has in fact been proposed in [1], it has been settled on a secure mathematical ground in [2] and supported by a lot of examples [3],[4]. In this paper we aim to discuss a possible extension to non-stationary cases.
We use the framework of classical field theory in its Lagrangian formulation; we require the theory to be natural, which means that each diffeomorphism of spacetime is a symmetry for the Lagrangian. For our purpose it is useful to rely on the geometric language of fiber bundles in which the calculus of variations is most naturally defined. In this framework, because of the first principle of thermodynamics, the product between the "temperature of the black hole horizon" and the "variation of entropy" is equal to an integral of a suitable $(n-2)$-form at space infinity (in a space

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time of dimension $n \geq 3$). The integrated form $\alpha(L, \sigma, \xi, X)$ is obtained as the variation of a suitable conserved quantity (in the sense of Nöther's theorem) and it is associated to a vector $\xi = \partial_t + \Omega\partial_\phi$ on spacetime. In the case of stationary black holes, it is possible to transform the integral at space infinity into an integral on a trapping surface Π for the singularities, since the form α turns out to be closed:

$$T\delta_X S = \int_\Pi \alpha(L, \sigma, \xi, X) \tag{1}$$

If the horizon of the black hole is bifurcate and we choose Π to coincide with the bifurcation surface, we obtain, as a particular case, the same formula obtained by R. Wald and V. Iyer in [1]. We stress however that the above assumptions on Π are unnecessary, often difficult to deal with and sometimes impossible (see ref.s [2], [3], [4]).

In this paper we try to extend the same definition of entropy to the case of non-stationary black holes. We here consider black holes with an oscillating horizon without a quadrupole momentum, so that they do not emit gravitational waves (see, for example, R. Wald and V. Iyer [1] or Frolov [11],[12]), for which the first principle of black holes thermodynamics has the form:

$$\delta_X M = T\delta_X \mathcal{S} + \Omega\delta_X J \tag{2}$$

because they can be considered as isolated systems.
The problems which occur in the non-stationary case are related to the fact that the vector ξ is no longer a Killing vector for the solution σ, a fact that in stationary case is extensively used to prove that the form α is closed (see ref.s [1], [2]). When we try to transform the integral at spatial infinity into an integral on a "finite surface", a volume integral of the so-called *symplectic form* $\omega(\sigma, \xi, X)$ appears because the form α is no longer closed, i.e. $Div\ \alpha(L, \sigma, \xi, X) \neq 0$. In this case the entropy is defined as the sum of two integrals:

$$T\delta_X S_{dyn} = \int_\Pi \alpha(L, \sigma, \xi, X) + \int_\Sigma \omega(\sigma, \xi, X) \tag{3}$$

where Σ is the volume of the region enclosed between Π and space infinity. We will show that both integrals are well defined and can be easily evaluated in the framework we use.
An earlier proposal for dynamic black hole entropy was given by R. Wald and V. Iyer [1]. They just tried to adapt the definition given for the stationary case but, in the end, their definition turned out not to be covariant due to unessential requirements made on the existence and structure of the horizons (see [2]). The new definition we propose satisfies all the conditions imposed by R. Wald and V. Iyer in [1] and in addition, as a consequence of the geometrical framework we use, our proposal is automatically covariant with respect to any fibered morphism, i.e. any redefinition of fields. This is a stringent requirement from a physical viewpoint, since any lack of covariance produces results which are either wrong or at least require a lot of efforts in order to get a correct physical interpretation.
A further problem, which we are not able to overcome, is due to the lack of examples on which one can test any prescription for entropy of non-stationary black holes. Basically we don't know any non-stationary and

geometrically well defined exact solution, for which one has a reasonable physical interpretation. For this reason we shall not present any example and direct application of our framework. Nevertheless we believe that the result we obtain is of interest, since it enlights the concept of entropy even in the stationary case by clariffing which are the fundamental properties of entropy and which are instead mere consequences of stationarity.
Furthermore, even if we do not know any explicit solution to test the formalism, we stress that the class of solutions that are under consideration is certainly not empty, physically relevant and, as mentioned above, it has been taken into account in the literature (see references [1], [11], [16]).

2 Notation and review of the stationary case

Hereafter we recall briefly the standard notation and the definition of entropy in the the stationary case (more details can be found in [2], [5], [8], [9]). Let us consider a configuration bundle B fibered on a spacetime M and let us denote by $J^k(B)$ its k-order jet bundle, i.e. the space where fields live together with their derivatives up to order k included. Fibered local coordinates on $J^k(B)$ are defined by $(x^\mu, y^i, y^i{}_\mu, ..., y^i{}_{\mu_1,...,\mu_k})$. Let us also denote by $\Lambda^n(T^*M)$ the bundle of n-forms over M. A *Lagrangian of order* k defined on B is a morphism of fiber bundles:

$$L : J^k(B) \to \Lambda^n(T^*M) \tag{4}$$

In the case of General Relativity in vacuum we can choose the second order Hilbert Lagrangian:

$$L = \frac{1}{2k} R\sqrt{g} ds \tag{5}$$

The variation of the generic Lagrangian (4) can be expressed through the so called *first variation formula*. We consider a vertical vector X on B and the variation of L along the flow of X, evaluated on a section σ of B:

$$< \delta L \circ j^k\sigma | j^k X >=< \mathbb{E}(L) \circ j^{2k}\sigma | X > + d[< \mathbb{F}(L,\gamma) \circ j^{2k-1}\sigma | j^{k-1}X >]$$

where $\mathbb{E}(L)$ and $\mathbb{F}(L,\gamma)$ are well-defined global morphisms. The Euler-Lagrange morphism:

$$\mathbb{E}(L) : J^{2k}(B) \to \Lambda^n(M) \otimes V^*(B) \tag{6}$$

is unique and it defines the field equations $\mathbb{E}(L) \circ j^{2k}\sigma = 0$. In this case $V^*(B)$ denotes the dual bundle of the vector bundle $V(B)$ of vertical vectors on B. The Poincaré-Cartan morphism $\mathbb{F}(L,\gamma)$ depends in general on the Lagrangian and on an arbitrary background connection γ on M:

$$\mathbb{F}(L,\gamma) : J^{2k-1}(B) \to \Lambda^{n-1}(M) \otimes V^*(J^{k-1}B) \tag{7}$$

In particular, the Euler-Lagrange morphism for General Relativity gives the vacuum Einstein equation

$$< \mathbb{E}(L) | X >= e_{\mu\nu}\delta g^{\mu\nu} ds = (R_{\mu\nu} - \frac{1}{2}\sqrt{g} g_{\mu\nu})\delta g^{\mu\nu} ds \tag{8}$$

while the expression of the Poincaré-Cartan morphism is in this case

$$< I\!\!F(L)|j^1X >= P^{\rho\theta}_{\alpha\beta}\,\nabla_\theta \delta g^{\beta\alpha}\,ds_\rho \tag{9}$$

where we have set

$$P^{\rho\theta}_{\alpha\beta} = -\left(\frac{1}{16\pi G}\right)\sqrt{g}\,[g^{\rho\theta}g_{\alpha\beta} - \delta^\rho_{(\alpha}\delta^\theta_{\beta)}] \tag{10}$$

We can notice the Poincaré-Cartan morphism (9) does not depend on any background connection due to the low order ($k = 2$) of the theory.
We recall also that one can introduce a definition for the Lie derivative of bundle sections with respect to the flow of a vector Ξ on B projectable over ξ on M as:

$$\pounds_\Xi\sigma = T\sigma(\xi) - \Xi\circ\sigma \tag{11}$$

A projectable vector field Ξ is an infinitesimal symmetry of the Lagrangian iff the following holds:

$$< \delta L\circ j^k\sigma \mid j^k\pounds_\Xi\sigma >= d(i_\xi L) \tag{12}$$

A bundle is *natural* iff for each spacetime diffeomorphism $f : M \to M$ on the basis M it is possible to find a canonical lift $\phi_f : B \to B$ on the bundle.
A field theory is natural iff the configuration bundle is natural and each diffeomorphism on the basis M is a symmetry of the Lagrangian L (in the sense that L is invariant under the pull back via any lift of ϕ_f). In natural theories we can define a Lie derivative with respect to a spacetime vector field ξ by setting:

$$\pounds_\xi\sigma = \pounds_\Xi\sigma \tag{13}$$

In this letter we will treat only natural theories, but the formalism introduced here is also valid for the more general case of gauge-natural theories [2], [10].

In the case of the Hilbert Lagrangian (5) which gives to General Relativity the structure of a natural theory, the covariance condition (12) with respect to the vector ξ can be expressed in the form:

$$d_\rho(\xi^\rho L) = \frac{1}{2}e_{\mu\nu}\pounds_\xi g^{\mu\nu} - \frac{1}{2k}\sqrt{g}g^{\alpha\beta}\pounds_\xi R_{\alpha\beta} \tag{14}$$

where $e_{\mu\nu}$ are the coefficients of the Euler-Lagrange morphism given by equation (8). From the first-variation formula and the covariance condition it is possible to formulate the Nöther's theorem which associates to any vector field ξ on the spacetime, a conserved current $\mathcal{E}(L,\xi)$ so that:

$$Div(\mathcal{E}(L,\xi)) = \mathcal{W}(L,\xi) \tag{15}$$

where Div denotes the (formal) divergence operator.
The *work-form* $\mathcal{W}(L,\xi)$ vanishes on-shell, i.e. along solutions of field equations. For General Relativity in vacuum we obtain that:

$$\mathcal{E}^\lambda(L,\xi) = d_\mu\Big[-\frac{1}{2k}\sqrt{g}(\nabla_*{}^\lambda\xi^\mu - \nabla_*{}^\mu\xi^\lambda)\Big] + (R_{\rho\nu} - \frac{1}{2}\sqrt{g}g_{\rho\nu})g^{\nu\lambda}\xi^\rho \tag{16}$$

For each natural theory (as well as for gauge-natural theories) using Bianchi's identities, it is possible to decompose the current $\mathcal{E}$ as:

$$\mathcal{E}(L,\xi) = \widetilde{\mathcal{E}}(L,\xi) + Div(\mathcal{U}(L,\xi)) \tag{17}$$

where $\widetilde{\mathcal{E}}$ is defined the *reduced current* and $\mathcal{U}$ is defined the *superpotential* of the theory. The reduced current $\widetilde{\mathcal{E}}$ vanishes on-shell. For the Lagrangian (5) the gravitational superpotential $\mathcal{U}$ can be explicitely calculated (see [5]) and it is known to be:

$$\mathcal{U}(L,\xi) = -\frac{1}{2k}\sqrt{g}(\nabla_*{}^{\lambda}\xi^{\mu} - \nabla_*{}^{\mu}\xi^{\lambda})ds_{\lambda\mu} \tag{18}$$

which is called the *Komar superpotential* [5]. The conserved quantities can now be obtained integrating the current $\mathcal{E}$ on a $(n-1)$-region D, i.e. a compact submanifold of M with a compact boundary ∂D. So the conserved quantities are integrals of the superpotential on ∂D. Generally the quantities obtained are not conserved with respect to the "time" defined by an ADM splitting of spacetime in spacelike surfaces. This happens, e.g., when the timelike vector ξ is a Killing vector for the solution g (see [18]). We stress that all the above quantities are linear with respect to the vector ξ (toghether with its derivatives).
However, if we calculate the conserved quantities for General Relativity integrating the superpotential (18) on ∂D, the mass obtained does not assume the physically expected value. This is the well known *anomalous factor problem* which affects the Komar superpotential. There are at least two different ways to solve the problem [5].
If we consider the variation of conserved quantities, defined above, this expression will suggest us to define the variation of the corrected conserved quantities by the ADM prescription [5]:

$$\delta_X \widetilde{Q}_D(L,\xi,\sigma) = \int_{\partial D}[\delta_X\mathcal{U}(L,\xi,\sigma) - i_\xi(< I\!F(L,\gamma)\circ j^{2k-1}\sigma | j^{k-1}X >)]$$

which gives us the expected quantities, of course up to an integration costant. To construct this formula in a covariant way it is necessary to introduce a *background connection* γ and the conserved quantities will depend on the background connection chosen. We can consider γ to provide us a "zero level" for the energy, so that (as it is physically resonable) it is like a parameter for the theory. In the case of General Relativity we can choose as a sort of "natural" background connection the Levi-Civita connection of any *background metric.*
On the other hand, one can notice that the Lagrangian (5) of General Relativity can be written as the sum of a first order Lagrangian and a divergence depending on the background metric. The first order Lagrangian gives us a mechanism analogous to the ADM formalism (thought explicitely covariant and independent on the choice of a foliation) to calculate the corrected conserved quantities [6]. The background fixing produces in this case an additional boundary term which solves the anomalus factor problem. The first order Lagrangian method is more general (in fact it is also applicable to non-compact solutions), while ADM formalism is not applicable in this case because space infinity is not a priori

asymptotically flat. This is particularly important to our purposes, since cosmological solutions are usually non-compact and have general a different asymptotical structure.

Mass and angular momentum for a black hole solution of Einstein equations are defined as the conserved quantities respectively connected to the vectors ∂_t and ∂_ϕ on spacetime. The corrected quantities, which provide us the physically expected values, are defined as integals at space infinity:

$$M = \int_\infty [\mathcal{U}_{Komar}(L, \partial_t, g) - B(L, \partial_t, g) \tag{19}$$

$$J = -\int_\infty [\mathcal{U}_{Komar}(L, \partial_\phi, g) - B(L, \partial_\phi, g)] \tag{20}$$

where the $(n-2)$-form B is defined through integration of the variational equation on ∂D:

$$\delta_X B(L, \xi, g) = i_\xi Div < I\!\!F(L, \gamma) \circ j^3 g | j^1 X > \tag{21}$$

see for example [2], [7].
A covariant (and somehow canonical) choice for vacuum General Relativity is:

$$B(L, \xi, g) = -\sqrt{g} g^{\alpha\beta} \xi^{[\lambda} w^{\mu]}_{\alpha\beta} ds_{\lambda\mu} \tag{22}$$

where we have set γ and Γ to be the Chistoffel's symbols for the metric and the background connection respectively and we have defined:

$$\begin{cases} w^\mu_{\alpha\beta} = u^\mu_{\alpha\beta} - U^\mu_{\alpha\beta} \\ U^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \Gamma^\rho_{\rho(\beta}\delta^\mu_{\alpha)} \\ u^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} - \gamma^\rho_{\rho(\beta}\delta^\mu_{\alpha)} \end{cases}$$

3 Definition of entropy in the stationary case

The entropy of a stationary black hole solution is defined as the macroscopical quantity which satisfies the first principle of thermodynamics (1). We impose that X is a solution of linearized field equations and both the temperature T and the angular velocity of black hole horizon Ω are constant parameters depending on the class of solutions chosen. The temperature T is just defined as the temperature of the Hawking radiation $T = \frac{\kappa}{2\pi}$, where κ is the surface gravity, as shown in [14], [18], [19], [20] by means of Euclidean path integrals. On the other hand Ω is defined so that $\mid \xi \mid^2$ vanishes on the BH horizon. If we solve (1) with respect to $\delta_X \mathcal{S}$ we obtain that:

$$\delta_X \mathcal{S} = \frac{1}{T}(\delta M - \Omega \delta J) == \frac{1}{T} \int_\infty [\delta_X \mathcal{U}(L, \xi, \sigma) - i_\xi < I\!\!F(L, \gamma) | j^1 X >]$$

where ∞ means the space infinity of a spacelike slice and $\xi = \partial_t + \Omega \partial_\phi$.

Under the hypotheses that ξ is a Killing vector for g and X is a solution of linearized field equations it is possible to prove in a very general framework (see [2]) that the quantity under integral:

$$\alpha(L, g, \xi, X) = \delta_X \mathcal{U}(L, \xi, g) - i_\xi < I\!\!F(L, \gamma)|j^1 X > \tag{23}$$

is a closed form. This allows us to redefine $\delta_X \mathcal{S}$ as an integral on any spatial surface which is homologically equivalent to ∞. In this definition we do not have any additional requirement about maximality of the solution considered neither about the horizon properties. In particular it is not necessary to require ξ to vanish on the trapping surface. This fact allows us to apply the definition to a wider range of solutions and simplifies both conceptually and computationally the calculations (see [2], [3], [4]).

If the solution admits a bifurcate Killing horizon and we can choose a bifurcation surface on which ξ vanishes, then our more general definition reproduces, as a very particular case, the one given by Wald and Iyer in [1]. This latter definition is not applicable to solutions for which ξ is not a Killing vector (non- stationary solutions, non-asymptotically flat solutions, etc...).

4 Variation of conserved quantities

It is possible to express a bundle morphism (for example the Euler-Lagrange and the Poincaré-Cartan morphisms which are k-forms on M) in local fibred coordinates. In this local formalism, any such morphism appears to be a linear combination of the vector field components (ξ^μ, ξ^i) together with their derivatives up to order r ($r = 1$ for the example of General Relativity under investigation). Let us thence consider a derivation δ (i.e. a linear operator which satisfies the Leibniz rule). If we are able to calculate the δ-derivative of a vector field component and of the whole n-form then, by applying the Leibniz rule, we are able to define the derivative of the coefficients of the linear combination.

In our case we apply this rule to the Lie derivative and to the variation along the flow of a vector field X, which are both derivations. For example, if we choose $\pounds_\xi$ as a particular derivation onto the Poincaré-Cartan morphism, where ξ is a vector field on spacetime and $X \in V(M)$, we obtain for a second order theory:

$$\pounds_\xi < I\!\!F(L, \gamma)|j^1 X >= \pounds_\xi \Big[p_i^\lambda X^i + p_i^{\lambda\mu} X_\mu^i \Big] ds_\lambda \tag{24}$$

and applying the Leibniz rule we obtain:

$$\begin{cases} \pounds_\xi p_i^\nu = \Big(d_\mu \xi^\mu p_i^\nu - d_\mu \xi^\nu p_i^\mu + \xi^\mu d_\mu p_i^\nu + \partial_i \xi^j p_j^\nu \Big) \\ \pounds_\xi p_i^{\nu\rho} = \Big(d_\mu \xi^\mu p_i^{\nu\rho} - d_\mu \xi^\nu p_i^{\mu\rho} + + \xi^\mu d_\mu p_i^{\nu\rho} + \partial_i \xi^j p_j^{\nu\rho} - d_\mu \xi^\rho p_i^{\nu\mu} \Big) \end{cases}$$

We remark that this definition for the Lie derivatives of the Poincaré-Cartan morphism can also be obtained by considering $p_i^\nu, p_i^{\nu\rho}$ as the local expressions of a section on a suitable fiber bundle and thence applying the general definition (11) of Lie derivatives of sections of fiber bundles

(see ref. [21], [22]).

It is now possible to analyze the divergence of the form $\alpha(L,\sigma,\xi,X)$ in the case that no conditions whatsoever are imposed on the solution σ. In the case of stationary black holes one requires ξ to be a Killing vector of the solution. Accordingly we assume in the general case that ξ is a symmetry for σ, i.e. $\pounds_\xi\sigma = 0$ which is fundamental to prove that $d\alpha(L,\sigma,\xi,X) = 0$ (see [2]).
If we relax this condition such a divergence does not vanish anylonger. This is the case of non-stationary black holes in a relativistic theory. In this general setting we have that:

$$\begin{aligned} Div(\alpha(L,\xi,\sigma,X)) &= Div(\delta_X\mathcal{U}(L,\xi) - i_\xi < \mathbb{F}(L,\gamma)|j^{k-1}X >) = \\ &= \delta_X < \mathbb{F}(L,\gamma)|j^{k-1}\pounds_\xi\sigma > -\pounds_\xi(< \mathbb{F}(L,\gamma) \mid j^{k-1}X >) \\ &- \delta_X\widetilde{\mathcal{E}}(L,\xi) - i_\xi < \mathbb{E}(L)|X > \end{aligned}$$

(see for example [2]).
Let us thence analyze each term of this expression. The Euler-Lagrange morphism vanishes on-shell $< \mathbb{E}(L)|X >= 0$. The variation of the reduced current can be expressed as:

$$Div\delta_X\widetilde{\mathcal{E}}(L,\xi) = -\delta_X < \mathbb{E}(L)|\pounds_\xi\sigma > \tag{25}$$

and this term is identically vanishing since X is a solution of the linearized field equation (see [2]).
Using the prescription given for the variation of fiber bundle morphisms, in the case of a theory of order $k = 2$, we can analyze the two terms left on the right hand side and we obtain as a special case:

$$\begin{aligned} Div \quad & [\alpha(L,\xi,\sigma,X)] = \omega(L,\xi,\sigma,X) = \\ = \quad & < \delta_X\mathbb{F}(L)|j^1\pounds_\xi\sigma > - < \pounds_\xi\mathbb{F}(L,\gamma)|j^1X > + < \mathbb{F}(L)|j^1Z > \end{aligned}$$

where ω is an $(n-1)$-form on spacetime and $Z = Z^i\partial_i$ is a vertical field defined as $Z^i = \Big(\partial_j X^i \pounds_\xi \sigma^j\Big)$ so that it lifts to $j^1Z = Z^i\partial_i + \Big(d_\mu Z^i\Big)\partial_i{}^\mu$.
Let us stress that in the above expression each term is "under control" in the sense that it can be analytically calculated whenever a Lagrangian is given for the theory. The expression of $Div[\alpha(L,\xi,\sigma,X)]$ is fundamental to our purpose; in fact it will contribute to the entropy formula under the form of a volume integral.
To summarize, in the case of under analysis (General Relativity in vacuum), we see that it is possible to calculate ω and α, using the formula for the Komar superpotential and for the Poincaré-Cartan morphism, expressed in local coordinates by (9), (18), namely in our case:

$$\begin{cases} \omega(L,\xi,\sigma,X) = \delta_X < \mathbb{F}(L,\gamma)|j^1\pounds_\xi g > -\pounds_\xi < \mathbb{F}(L,\gamma)|j^1X > \\ \alpha(L,\xi,\sigma,X) = \delta_X\mathcal{U}(L,\xi,g) - i_\xi < \mathbb{F}(L,\gamma)\circ j^3g|j^1X > \end{cases}$$

where variations and Lie derivatives can also be defined according to (24) as usual for differential forms.

5 Non-stationary black holes

In this section we extend our definition for black hole entropy to the case of non-stationary black holes. As a motivation let us mention that cosmological solutions of Einstein's equations are usually not stationary and not asymptotically flat models, i.e. the solution does not admit any timelike Killing vector. In this case the $(n-2)$-form α is no longer closed and we cannot easily define the entropy as a boundary integral on a trapping surface for the singularity.
In our model the first principle of thermodynamics is the same used for stationary black holes. We consider black holes which do not emit gravitational waves, so we consider only solutions without a quadrupole momentum [1], [11], [12]. This means that the system is isolated and electrically not charged. The geometrical formalism we use is manifestly covariant. We will also show that our proposal satisfies all the reliability conditions stated by Wald and Iyer in [1].
We define again the entropy for a non-stationary solution of Einstein equations as the macroscopical quantity which satisfies the first principle of thermodynamics (1). The definition is the same given before for the case of stationary black holes. In this new case, however, T and Ω cannot be calculated as the temperature and the angular velocity of black hole horizon, but they can be considered as a priori parameters of the theory. The only requirement is to ask these parameters to realize an integrable first principle of thermodynamics. The choice among them has to be carried over on the basis of some external physical consideration. However this is not a feature of non-stationary solutions; even in the stationary case, if we choose quasi-local energy instead of mass we obtain a different (but integrable) first principle [18]. This fact will be subject of further investigations.
On the other hand, the mass and the angular momentum may no longer be time-conserved on a spacelike ADM slice of spacetime, but they are covariantly conserved, i.e. they obey a continuity equation; in other words they are conserved in the sense of Nöther theorem even if their values may change in time.
We can substitute in the first principle the expressions (19) and (20) for mass and angular momentum calculated by means of Nöther theorem and we will obtain an expression which defines the variation of entropy as an integral on space infinity (23). Now, in the case of non-stationary black holes, we have to take into account that α is not closed to evaluate the same quantity on a trapping surface. We have to consider the form ω and its integral on a volume Σ *between* the trapping surface Π and the space infinity. So we will obtain the formula for entropy under the form:

$$T\delta_X S_{dyn} = \int_{\Pi} \alpha(L, g, \xi) + \int_{\Sigma} \omega(L, g, \xi) \tag{26}$$

where T is the black hole temperature and the integrated forms ω and α are defined in the previous section. In this formula each term is explicitly known and computable once the Lagrangian and the exact solution g are specified. The only restriction on the theories we analyze is the fact that they have to be well-defined from a Lagrangian viewpoint.

The effort to apply this formula to known solutions has been vain up to now. In the case of General Relativity in vacuum, in fact, to our knowledge there are no well defined exact non-stationary solutions in literature, even if the Hilbert Lagrangian which defines the theory is global and covariant. If we consider otherwise the case of General Relativity in interaction with matter it is then possible to find in literature some explicit non-stationary solution; in this case it would be easily possible to generalize the definition of entropy (26) to treat also these theories by just adding an interaction term to the superpotential which eventually enters the final formula (26) (see [13]). In this latter case, however, there is no well defined global Lagrangian for the theory, because of the exhotic properties of the gas matter considered, which is an essential requirement to calculate conserved quantities in a geometrical framework. However we carry over a theoretical analysis of the case of General Realtivity in vacuum.

Wald and Iyer imposed some conditions *a priori* on the reliability of the definition which our definition satisfies by default:
• In the case of stationary black holes we must have $T\delta_X\mathcal{S} = T\delta_X\mathcal{S}_{dyn}$. To show this fact it is enough to say that when the solution is stationary we have $\pounds_\xi g = 0 \Rightarrow \omega(L, g, \xi) = 0$.
• We have to show that in the case of non-stationary perturbations, generated by a field $\widetilde{X}$, of a stationary solution we have $T\delta_{\widetilde{X}}S_{dyn} = T\delta_{\widetilde{X}}S$. In this case once again $\omega(L, g, \xi) = 0$. This is related to the fact that what we need is $\pounds_\xi g_0 = 0$ where g_0 is the unperturbed solution, which is stationary.
• The entropy for a theory defined by an equivalent Lagrangian $L + Div\theta$ should be the same calculated for the theory defined by the Lagrangian L. It easy to see that for any pure divergence Lagrangian we have $\alpha(Div\theta, \sigma, \xi) = 0$. Since α is linear in L, in fact we have that $\alpha = 0 \Rightarrow \omega = 0$ because we have chosen a pure divergence Lagrangian.
• Our definition must satisfy the second principle. This point is left out to future investigations, but we stress that at the moment the problem is still out of control even in the stationary case [1]; however it is reasonable to say that the second principle is related to the second variation of the fibered morphisms we have constructed, and thence to the positivity of energy.
• Finally, our definition should be covariant under field redefinition. Our formalism is manifestly covariant by construction and our definition satisfies this condition too. It is easy to prove this claim if we consider the trasformation rules for the Poincaré-Cartan morphism coefficients and for the field $j^{k-1}X$.

6 Conclusions and perspectives

We have proposed a new prescription to calculate the entropy for a non-stationary black hole. This formula is applicable to a well defined relativistic theory with a known (global) Lagrangian and whenever Lie derivatives are well-defined so that we can define covariant conserved quantities at

space infinity. The application of our formalism to calculate entropy for a well-defined explicit solution is immediate; once we have the solution in a local coordinate system it is possible to apply the algorithmical formalism we have developed to calculate the conserved quantities and entropy. The formula we have proposed is independent on the choice of the trapping horizon for the singularity; if we consider horizons belonging to the same homotopy class, the result obtained for entropy is invariant (see for example the TAUB-BOLT solution, a discussion of which is given in [4]). From a physical viewpoint this definition satisfies the conditions stated by Wald and Iyer in [1] and in particular it is covariant. A future task will be to calculate explicitly the entropy for some exact non-stationary solution of Einstein equations. We will furthermore investigate the second principle for our definition both for stationary and non-stationary black holes.

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